

The Ground Axiom

JOEL DAVID HAMKINS

Many interesting models of set theory are not obtainable by nontrivial forcing over an inner model. This includes, for example, the constructible universe L , the canonical model $L[\mu]$ of a measurable cardinal and many instances of the core model K (although Schindler has observed that the least inner model M_1 of a Woodin cardinal actually is a nontrivial forcing extension of an inner model). To highlight this phenomenon, my student Jonas Reitz and I introduced the Ground Axiom, which asserts that the universe is not a set forcing extension of any proper inner model.

Ground Axiom (H, Reitz). *The universe is not a forcing extension of any inner model by nontrivial set forcing. Specifically, if $W \subsetneq V$ is a transitive inner model of ZFC and $G \subseteq \mathbb{P} \in W$ is W -generic, then $V \neq W[G]$.*

Despite the *prima facie* second order nature of this assertion, the Ground Axiom is actually first order expressible in the language of set theory.

Theorem 1 (Reitz, Woodin). *The Ground Axiom is first order expressible in the language of ZFC.*

This theorem is the starting point of Reitz's dissertation [Rei], but an essentially equivalent assertion was observed independently by Woodin [Woo]. Reitz's proof makes use of ideas arising in Laver's [Lav] recent result that a ground model is always definable in its forcing extensions.

Theorem 2 (Laver). *If $V \subseteq V[G]$ is a set forcing extension, then V is a definable class in $V[G]$, using parameters in V .*

This result was also observed independently by Woodin [Woo]. Laver's proof is connected with my recent theorem showing the extent to which embeddings in a forcing extension must be lifts of ground model embeddings.

Key Definition 3.

- (1) $V \subseteq V[G]$ exhibits δ -covering if every set of ordinals in $V[G]$ of size less than δ is covered by a set of size less than δ in V .
- (2) $V \subseteq V[G]$ exhibits δ -approximation if whenever $A \in V[G]$, $A \subseteq V$ and $A \cap a \in V$ for all $a \in V$ with $|a|^V < \delta$, then $A \in V$.

Such forcing extensions are abundant in the large cardinal literature. Any forcing notion of size less than δ has δ -approximation and δ -covering. More generally, any forcing of the form $\mathbb{P} * \dot{\mathbb{Q}}$, where \mathbb{P} is nontrivial, $|\mathbb{P}| < \delta$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $<\delta$ -strategically closed, exhibits δ -approximation and δ -covering. Therefore, such forcing as the Laver preparation or the canonical forcing of the GCH exhibit approximation and covering for many values of δ .

A special case of the main theorem of [Ham03] is:

Theorem 4. *If $V \subseteq V[G]$ exhibits δ -approximation and δ -covering, then every ultrapower embedding $j : V[G] \rightarrow M[j(G)]$ above δ in $V[G]$ is the lift of an embedding $j \restriction V : V \rightarrow M$ definable in V .*

In particular, $M \subseteq V$ and $j \restriction A \in V$ for all $A \in V$. The full theorem applies to all sufficiently closed embeddings, including many types of extender embeddings. The general conclusion is that extensions with δ -approximation and δ -covering have no new large cardinals above δ . The proofs of Theorems 2 and 4 make similar and extensive iterated use of the approximation and cover properties in their arguments that the respective classes are definable.

Returning to the Ground Axiom, one observes that the natural models of GA, such as L and $L[\mu]$, exhibit the GCH and many other regularity features. Are these a consequence of the Ground Axiom? The answer is no.

Theorem 5 (Reitz). *If ZFC is consistent, then $\text{ZFC} + \text{GA} + \neg\text{CH}$ is consistent.*

The method is flexible and shows that if σ is any Σ_2 assertion consistent with ZFC, then $\text{ZFC} + \text{GA} + \sigma$ is consistent. These theorems are proved by forcing, which is a bit paradoxical as GA asserts that the universe is not a forcing extension. Specifically, resolving the paradox, they are proved by *class* forcing. Using McAloon's [McA71] methods to force strong versions of $V = \text{HOD}$, one codes the universe into the continuum function, and then GA holds with any desired V_α left intact. The hypothesis $V = \text{HOD}$, however, by itself does not imply GA. Conversely, at the Set Theory Workshop at the Mathematisches Forschungsinstitut Oberwolfach (0549, December 4-10, 2005), Woodin suggested a very promising line of argument to show that the Ground Axiom is consistent with $V \neq \text{HOD}$, which is now being investigated. Reitz has proved, using large cardinal indestructibility results, that the Ground Axiom is consistent with nearly any kind of large cardinal, from measurable to strong to supercompact and beyond.

Theorem 6. *If the existence of a supercompact cardinal is consistent with ZFC, then it is consistent with $\text{ZFC} + \text{GCH} + \text{GA}$.*

These theorems fit very well into the long-standing set theoretic program, advanced by Woodin and others, to obtain the features of the canonical inner models of large cardinals, but to obtain them by forcing over arbitrary models of those large cardinals. The Ground Axiom is such a feature.

Ordinarily, one imagines forcing as a way to reach out into larger mathematical universes. Here, however, we are reaching from a given universe down into the possible ground models of which it is a forcing extension. Given a model of set theory, perhaps we can strip away a top layer of forcing and be left with a ground model, a bedrock model if you will, that is not itself obtainable by forcing from any smaller inner model. In this case, the original universe satisfies:

Bedrock Axiom. *The universe V is a set forcing extension $V = W[G]$ of an inner model W of $\text{ZFC} + \text{GA}$.*

The model W is a bedrock model for V in the sense that it is a minimal ground model for V , having no ground model below it. This axiom is first order expressible

for the same reasons that the Ground Axiom was. Since $V = W$ is allowed, we have $\text{GA} \implies \text{BA}$. A common feature of the models of GA and their forcing extensions, of course, is that they are all forcing extensions of a model of GA , and hence themselves models of BA . Are there any other models? Yes.

Theorem 7 (Reitz). *If ZFC is consistent, then $\text{ZFC} + \neg\text{BA}$ is consistent. Indeed, if σ is any Σ_2 assertion consistent with ZFC, then $\text{ZFC} + \text{BA} + \sigma$ is consistent.*

Perhaps the main open question here is:

Question 8. *Is the bedrock model unique when it exists?*

Several attacks on this question were suggested by various participants at the Oberwolfach workshop, and a promising investigation has now ensued.

The theme of current work is to investigate the spectrum of possible ground models of the universe, the spectrum of inner models W of which the universe V is a forcing extension $V = W[G]$. The results above provide a uniform definition for these ground models W in V . By varying the parameters in this definition, one obtains in effect a class enumeration of the possible ground models W for V . That is, the class I of parameters p giving rise to a ground model W_p such that V is a forcing extension $V = W_p[G_p]$ is definable, and the corresponding meta-class $\{W_p \mid p \in I\}$ of possible ground models is in effect definable as $\{\langle p, x \rangle \mid x \in W_p \text{ \& } p \in I\}$. Thus, the treatment of the spectrum of possible ground models is entirely a first order affair of ZFC. Another theme is to restrict attention to a particular class of forcing notions, with such axioms as GA_{ccc} , which asserts that the universe is not a nontrivial forcing extension of an inner model by c.c.c. forcing. We can produce models, for example, of $\neg\text{GA} + \text{GA}_{\text{ccc}} + \sigma$, for any consistent Σ_2 assertion σ ; these models are forcing extensions of an inner model, but are not obtainable by c.c.c. forcing. Similar questions and results abound here.

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THE COLLEGE OF STATEN ISLAND OF THE CITY UNIVERSITY OF NEW YORK, MATHEMATICS,
2800 VICTORY BOULEVARD, STATEN ISLAND, NY 10314, USA

THE GRADUATE CENTER OF THE CITY UNIVERSITY OF NEW YORK, MATHEMATICS, 365 FIFTH
AVENUE, NEW YORK, NY 10016, USA

E-mail address: jhamkins@gc.cuny.edu, <http://jdh.hamkins.org>